

A New Class of Nonsingular Exact Solutions for Laplacian Pattern Formation

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Abstract

We present a new class of exact solutions for the so-called *Laplacian Growth Equation* describing the zero-surface-tension limit of a variety of 2D pattern formation problems. Contrary to the common belief, we prove that these solutions are free of finite-time singularities (cusps) for quite general initial conditions and may well describe real fingering instabilities. At long times the interface consists of N separated moving Saffman-Taylor fingers, with “stagnation points” in between, in agreement with numerous observations. This evolution resembles the N -soliton solution of classical integrable PDE’s.

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The problem of pattern formation is one of the most rapidly developing branches of nonlinear science today.[1] Of special interest is the study of the front dynamics between two phases (interface) that arises in a variety of nonequilibrium physical systems. If, as it usually happens, the motion of the interface is slow in comparison with the processes that take place in the bulk of both phases (such as heat-transfer, diffusion, etc.), the scalar field governing the evolution of the interface is a harmonic function. It looks natural then, to call the whole process *Laplacian growth*. Depending on the system, this harmonic scalar field is a temperature (in the freezing of a liquid or Stefan problem), a concentration (in solidification from a supersaturated solution), an electrostatic potential (in electrodeposition), a pressure (in flows through porous media), a probability (in diffusion-limited aggregation), etc.

We present in this paper a new class of solutions of the 2D-Laplacian growth problem in the limit of zero surface tension. These solutions are quite general because no symmetries of the moving interface are assumed. Most remarkably, they do not develop finite time singularities but, contrary to the common belief, remain smooth for all finite times. Thus, they may describe real fingering instabilities when surface tension is very small. In the long-time limit they give rise to N separated fingers, each of which (for enough separation) describes the Saffman-Taylor finger [2] in channel geometry, and whose evolution closely resembles the N -soliton formation in nonlinear integrable PDE's.

In the absence of surface tension, whose effect is to stabilize the short-wave perturbations of the interface, the problem of 2D Laplacian growth is described as follows:

$$(\partial_x^2 + \partial_y^2)u = 0, \tag{1}$$

$$u|_{\Gamma(t)} = 0, \tag{2}$$

$$\partial_n u|_{\Sigma} = 1, \tag{3}$$

$$v_n = -\partial_n u|_{\Gamma(t)}. \tag{4}$$

Here $u(x, y; t)$ is the scalar field mentioned above, $\Gamma(t)$ is the moving interface, Σ is a fixed external boundary, ∂_n is the component of the gradient normal to the interface (i.e. the normal derivative), and v_n is the normal component of the velocity of the front.

We consider in this paper an infinitely long interface. We introduce then a time-dependent conformal map f from the lower half of a “mathematical” plane, $\zeta \equiv \xi + i\eta$, to the domain of the physical plane, $z \equiv x + iy$, where the Laplace equation (1) is defined, $\zeta \xrightarrow{f} z$. We also require that $f(t, \zeta) \approx \zeta$ for $\zeta \rightarrow \xi - i\infty$. With this definition, the function $z = f(t, \xi)$ describes the moving interface.

Using this conformal map and taking into account the boundary conditions of the problem we find:

$$v_n = \frac{\text{Im}(\bar{f}_t f_\xi)}{|f_\xi|}, \quad (5)$$

and

$$-\partial_n u_{\Gamma(t)} = -\partial_l \psi = \frac{\partial \xi}{|\partial f|} = \frac{1}{|f_\xi|}, \quad (6)$$

where the overbar means complex conjugate, the subscripts t and ξ indicate partial derivatives with respect to t and ξ , respectively, ∂_l is the component of the gradient tangent to the interface, ψ is a harmonic function of x and y , conjugate to u , that satisfies $\psi = -\xi$ due to the boundary conditions (2) and (3). If we equate Eqs. (5) and (6), in accordance with Eq.(4), we finally obtain:

$$\text{Im}(\bar{f}_t f_\xi) = 1 \quad (7)$$

As in Ref. 3, we will refer to Eq. (7) as the *Laplacian Growth Equation* (LGE), because the scalar field determining the growth obeys the Laplace equation (1). The LGE was first derived, to our knowledge, in 1944 independently by Polubarinova-Kochina[4] and Galin[5]. This equation has remarkable properties, unexpected for nonlinear PDE’s, such as an infinite

set of conserved quantities[6], a pole decomposition[7], an exact solution in the class of finite polynomials[8] and other exact solutions (though for particularly symmetrical initial shapes)[9]. Also the Saffman-Taylor finger [2] is a particular travelling wave-solution of this equation. All these properties (except the latter one) are nontrivial and nonperturbative due to the nonlinear nature of the LGE.

Unfortunately, despite these remarkable properties, practically all known solutions of the LGE show finite-time singularities via the formation of cusps [7], [8], [10]. Therefore, all these nonperturbative results are helpless to shed light on the physics and geometry of the system in the long-term limit. (Although a few exact results have been presented that have no finite-time cusps [9], they correspond to cases with very restricted symmetries of the moving interface.) As a result it has been generally assumed that these finite time singularities are an essential feature of LGE solutions [11] and that, in this sense, the physics represented by the LGE is incomplete. Thus, the natural attitude was to include surface tension in the theory to stabilize the moving interface and get rid of the finite-time singularities.[11]

The main result of this paper is to show that, contrary to this widespread view, the LGE (i.e. a zero-surface tension limit of the 2D Laplacian pattern formation) admits quite a broad class of exact time-dependent solutions which remain smooth for an infinite time. These solutions are of “ N -finger type” because they lead to the formation of N well-developed and separated fingers in the long-time asymptotics (see Fig.1). Thus, the problem of finite-time cusps can be solved within the framework of the LGE. (However, there are problems in which the inclusion of surface tension is indeed unavoidable, e.g. the selection mechanism in dendritic growth).[1]

To introduce this new class of solutions, we start from the statement that any function $f(t, \zeta)$ whose derivative, f_ζ , has an arbitrary distribution of moving poles, $\zeta_k(t) \equiv \xi_k + i\eta_k$, and roots, $Z_k(t)$, in the upper-half plane, $\text{Im } \zeta > 0$, and no other singularities, is a solution of the LGE. This is easy to

verify by substitution of any such function f into Eq. (7). Let us consider the arbitrary case when f_ζ has $N + 1$ simple poles. Since we assume that

$$f_\zeta = \prod_{k=1}^{N+1} \frac{\zeta - Z_k(t)}{\zeta - \zeta_k(t)} \quad (8)$$

with all poles and roots initially lying above the real axis, we have

$$f = A(t) + \zeta - i \sum_{k=1}^{N+1} \alpha_k \log(\zeta - \zeta_k) \quad (9)$$

where $A(t)$ is an unknown function of time, and the factor $-i$ is taken just for convenience.

By substitution of the last equation for ζ real, into the LGE (Eq. (7)), we find that

- *i)* $A(t) = -it$;
- *ii)* $\alpha_k \in \mathbf{C}$ does not depend on time ;
- *iii)* the time-dependence of the poles ζ_k is governed by the following constants of motion β_k :

$$\beta_k \equiv f(\bar{\zeta}_k) = -it + \bar{\zeta}_k - i \sum_{\ell=1}^{N+1} \alpha_\ell \log(\bar{\zeta}_k - \zeta_\ell), \quad 1 \leq k \leq N + 1. \quad (10)$$

Therefore, the motion of the interface is determined by Eqs. (9)-(10) for real ζ . This solution, for $N = 1$ corresponds to the development of an isolated finger, similar to the one found by Saffman for channel geometry. [12]

The break of analyticity of the interface (a cusp) occurs when at least one of the moving poles, $\zeta_k(t)$, or zeros, $Z_k(t)$, of f_ζ crosses the real axis, $\eta = 0$, of the mathematical plane, ζ . If all ζ_k 's and Z_k 's remain on the upper-half plane during the whole evolution, then the moving interface remains smooth (analytic) for an infinite time. To obtain sufficient conditions under which this is true for f given by Eqs. (9)-(10), we note the following:

(i) In order for the solution to exist as $t \rightarrow \infty$ and satisfy $\eta_k > 0$ for all finite times, all α_k 's must have positive real part. This is the only way that the divergent term $-it$ in the R.H.S. of Eq. (10) can be compensated. The term that compensates it is

$$-i\text{Re}(\alpha_k) \log(\bar{\zeta}_k - \zeta_k) = -i\text{Re}(\alpha_k) \log(-2i\eta_k) \quad (11)$$

and implies that $\eta_k \rightarrow 0$ as $t \rightarrow \infty$.

(ii) An isolated singularity, ζ_k , can never reach the real axis at a finite time. If it did, the term $-i\alpha_k \log(\bar{\zeta}_k - \zeta_k)$ in Eq. (10) would diverge and could not be compensated by any other. On the other hand, if all $\text{Re}\alpha_k$'s are positive, then groups of $M \leq N+1$ singularities could not reach the real axis simultaneously at a finite time for exactly the same reason. Therefore, we can conclude that, if $\text{Re}\alpha_k > 0$ and $\eta_k(t=0) > 0$ for $1 \leq k \leq N+1$, then $\eta_k(t) > 0$ for all finite times t and $\eta_k \rightarrow 0$ as $t \rightarrow \infty$.

(iii) We assume first that all α_k 's are real and positive. We calculate the derivative of f with respect to $\zeta \equiv \xi + i\eta$. After a little algebra, we write its real part as

$$\text{Re } f_\zeta = 1 + \sum_{k=1}^{N+1} \frac{\alpha_k \eta_k}{|\zeta - \zeta_k|^2} - \eta \sum_{k=1}^{N+1} \frac{\alpha_k}{|\zeta - \zeta_k|^2} \quad (12)$$

Since all α_k 's and all $\eta_k(t=0)$'s are positive, then by the result in (ii), also $\eta_k > 0$ for all finite times. Thus, we see from Eq. (12), that $\text{Re } f_\zeta$ equals zero only if η is strictly positive. This means that the zeros, Z_k , of f_ζ lie always on the upper-half of the mathematical plane and never cross the real axis.

Consequently, *if at time $t = 0$, all α_k 's are real and positive and all η_k 's are positive, then the interface represented by Eq.(9) remains smooth throughout its evolution.*

For the general case of complex α_k 's (with positive real part) we are currently unable to prove neither the existence nor the absence of finite-time blow-ups. (For $N = 0$ we can easily show that the solution (9) remains smooth for all time). In spite of the lack of a rigorous proof, we believe that

also in this case, there are no finite-time cusps for a broad range of initial conditions. This conjecture is supported by a series of numerical experiments that we will discuss in a forthcoming paper. [13]

Eq. (9) is not the only solution of the LGE that is characterized by the motion of simple poles. For example, if in Eq. (9) we replace $\log(\zeta - \zeta_k)$ by $\log(e^{i\zeta} - e^{i\zeta_k})$, and introduce a new parameter λ , we find a 2π -periodic solution, relevant for channel geometry, of the form:

$$f = -i\frac{t}{\lambda} + \lambda\zeta - i\sum_{k=1}^N \alpha_k \log(e^{i\zeta} - e^{i\zeta_k}) \quad -\pi \leq \text{Re } \zeta \leq \pi. \quad (13)$$

This solution must satisfy $\sum_k \alpha_k = 1 - \lambda$ so that $f_\zeta = 1$ for $\zeta \rightarrow -i\infty$. The parameter λ is the fraction of width of channel occupied by the fingers. This solution has the same properties as Eq. (9). In particular, there exist N constants of motion defined by $\beta_k \equiv f(\bar{\zeta}_k)$ and cusps are absent if all α_k 's are real and positive and all η_k 's are positive, as it follows from the equation:

$$f_\zeta = \lambda + \sum_{k=1}^N \frac{\alpha_k}{|1 - e^{i(\zeta_k - \zeta)}|^2} (1 - e^{-\eta_k} e^{i(\zeta - \xi_k)}) \neq 0 \quad \text{if } \text{Im } \zeta \leq 0. \quad (14)$$

Eq. (13), but with $e^{-i\zeta}$ instead of $e^{i\zeta}$, was proposed as a solution of the LGE in [11], where cusps were found via numerical simulations. Taking advantage of the corresponding constants of motion, β_k , we can easily show the necessity of these cusps if $\text{Re}(\alpha_k) > 0$ (as is the case in [11], where $\alpha_k = 1$ for all k).

The constants α_k have a clear geometrical meaning in the physical plane for both solutions (9) and (13): $|\pi\alpha_k|$ is related to the width of the gap between adjacent moving fingers (in the case of enough separation), while $\arg(\alpha_k)$ is related to the angle that this gap forms with the horizontal.[13] We show this property in Fig. 1, where we have plotted the interface $Y \equiv \text{Im } f(t, x)$ vs. $X \equiv \text{Re } f(t, x)$, for the solution (9) with two singularities ($N = 1$), at a particular time. The real parts of both α 's are positive, but while α_1 is purely real ($\alpha_1 = 0.8$), α_2 has a nonvanishing imaginary part ($\alpha_2 =$

$0.8 + i0.1$). We have drawn on top of each gap a dashed line of length $|\pi\alpha_k|$ and slope $\text{Im}(\alpha_k)/\text{Re}(\alpha_k)$ that highlights the meaning of α_k . We conclude then that, if all α_k 's are real and positive, the interface is a single-valued function $Y(X)$. Among the experiments on two-dimensional viscous fingering in a channel, one can find different degrees of bending and ramification of the moving fingers.[14] It follows from these experiments that non-single-valued interfaces generally appear, although with small bending. Therefore, complex α 's (possibly with small imaginary parts) would be necessary to describe them using Eq. (9). As we mentioned before, we expect that even in this more general setting, Eq. (9) still provides a meaningful (non-singular) solution, especially, if the imaginary parts of α 's are small enough.

The constants β_k also have a clear geometrical meaning in the physical plane: the points $(\text{Re } \beta_k, \text{Im } \beta_k + \text{Re } \alpha_k \log 2)$ are the coordinates of the tips of the gaps between fingers. We show this property in Fig. 2, where we have plotted the interface obtained from Eq. (9) at different times for a particular choice of $N + 1 = 7$ singularities and real and positive α_k 's. For the sake of generality we have deliberately taken the initial condition without any particular symmetry. We have indicated in this figure the location of the tips with asterisks. As one can see, they are “stagnation points” of the interface. This kind of stagnation points have been observed in numerous experiments [14] and numerical simulations [15]. Because all real experiments deal with non-zero surface tension, the experimental evidence of stagnation points in viscous fingering supports our conjecture that a small surface tension does not destroy the behavior of the zero-surface-tension solutions. It is interesting to note that we can also identify stagnation points occurring in numerical [16] and physical [17] experiments of diffusion-limited aggregation (DLA), (a feature usually explained as a “screening” effect). This is in accordance with the view (which we believe) that DLA fractal growth is described by the LGE in the continuous limit.[18]

Finally, we show in Fig. 3 the motion of the singularities in the mathe-

mational plane that correspond to the solution depicted in Fig. 2. Comparing both figures we can see that, while the singularities tend to the real axis, the interface takes the shape of a set of more and more well-developed and separated N fingers. The evolution of this interface resembles the N -soliton solution of classical exactly solvable PDE's, such as the Korteweg-de Vries or nonlinear Schrödinger equations, where in the long-time asymptotics one can also have N separated solitons, each of which described by the single soliton solution of the corresponding PDE. An evident difference is that fingers, unlike “classical” solitons, always have a non-zero velocity component normal to the interface. The connection between the N -soliton and the N -finger solutions is deeper than a superficial resemblance, as we will show in a forthcoming paper.

The separation of the singularities in the long-term asymptotics and the similarities with the N -soliton solutions suggests that there might be some transformation into action-angle variables in which this separation arises naturally. We also suspect that some Hamiltonian structure (at least formal) might underlie the whole evolution. We are presently working in this direction. The nontrivial extension of this work to the case of a closed (finite) interface is also of interest. In this case, our proof regarding the absence of finite-time singularities does not hold, however, the N -finger like solution with its associated constants of motion is still valid. The observation in physical and numerical experiments of very similar behavior to the infinite interface, including the existence of “stagnation points” [19], makes it seem likely that the class of solutions studied in this paper may also shed light on the problem of radial geometry.

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Figure Captions

Figure 1. We plot the interface, $Y \equiv \text{Im } f(t, \xi)$ as a function of $X \equiv \text{Re } f(t, \xi)$, at a particular time. for f defined in Eq. (9) with $A = -it$ and two singularities. $\alpha_1 = 0.8$; $\alpha_2 = 0.8 + i0.1$. The dashed lines that we have drawn on the gaps between fingers have length $|\pi\alpha_k|$ and slope $\text{Im}(\alpha_k)/\text{Re}(\alpha_k)$.

Figure 2. We plot the interfaces, Y as a function of X , at times $t = 0$, $t = 9$, $t = 24$, and $t = 30$, for f given by Eq. (9) with $A(t) = -it$, ζ real and $N + 1 = 7$. The evolution of the seven singularities was obtained using a fourth-order Runge-Kutta integrator to solve the set of evolution equations that are obtained by differentiating with respect to time Eq. (9) with $A(t) = -it$. The parameters of the simulations are $\alpha_1 = 0.80$, $\beta_1 = 6.00 - i37.11$, $\alpha_2 = 2.00$, $\beta_2 = 20.44 - i33.98$, $\alpha_3 = 1.00$, $\beta_3 = 35.61 - i34.49$, $\alpha_4 = 0.50$, $\beta_4 = 43.15 - i36.31$, $\alpha_5 = 1.50$, $\beta_5 = 54.58 - i32.78$, $\alpha_6 = 0.35$, $\beta_6 = 64.65 - i40.88$, $\alpha_7 = 1.80$, and $\beta_7 = 72.90 - i37.40$. We see that, from an initially bell-shaped form, the interface evolves into a situation characterized by $N = 6$ well separated fingers. We also observe the existence of “stagnation points” of the interface (indicated with asterisks).

Figure 3. We plot the trajectories of the moving singularities, $\zeta_k(t)$, on the mathematical plane, that correspond to the physical process depicted in Fig. 2. The asterisks indicate, from top to bottom, the points on the trajectory at times $t = 0$, $t = 9$, $t = 15$, and $t = 30$. We see that all the singularities go asymptotically in time to different values and that all $\eta \rightarrow 0$ as $t \rightarrow \infty$.

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